Quantum Optics
Quantum Electrodynamics
Course 1

Ref. Quantum Optics (Walls & Milburn)

Historical account of <u>Quantum-Optics & Quantum Electrodynamics</u>

1900 Planck discovered that spectral distribution of thermal light can be derived from postulating energy of an H.O. is quantized

1905: Photoelectric effect could be explained by quantized energy states of light.

1909 Taylor attempted to find quantum effects of light via optical interference using single photons. (Young exp)

1954 Hanbury-Brown-Twiss experiment → measurement of two-time intensity autocorrelation functions.
→ Photon bunching

1960 – LASER

1963: Glauber quantum theory of coherence → predicted
→ anti-bunching of light
Classical theory would require NEGATIVE probabilities

1975/76: prediction and measurement of non-classical light in two level system (resonance fluorescence)
→ QUANTUM OPTICS.

Example from today: future LIGO (Laser-interferometer gravitational wave observatory) used SQUEEZED LIGHT.
(first generated 1985 at Bell labs).

" The existance of a ZP Energy of size $h\nu/2$ is probable" – EINSTEIN (1913)

Examples of <u>Quantum Electrodynamics</u>: (QED) → vacuum processes

Key development (1) Lamb Shift 1952 in Hydrogen
Dirac Theory predicted $E_{2p1/2} = E_{2s1/2}$
→ non-relativistic effect due to vacuum

(2) Equation of State $(p + a/v^2)(v - b) = RT$
$V(r) \propto -\frac{3\hbar\omega_0\alpha^2}{4r^6}$ VAN DER WAALS

(3) Weisskopf–Wigner Theory: spontaneous emission (1930)

(4) Casimir force (1948) – force on metallic plates.

(5) Fundamental laser linewidth

(6) 1912 Blackbody Expression PLANCK $U = \frac{h\nu}{e^{h\nu/k_BT} - 1} + \frac{h\nu}{2}$

Quantum Optics : I
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Quick review: Quantized Harmonic Oscillator (H.O)

Classical: $$\boxed{E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m\omega^2 x^2 = \frac{1}{2} m p^2 + \frac{1}{2} m\omega^2 x^2}$$

were the speed is $\dot{x} = \frac{d}{dt} x$ and $x(t)$ the coordinate

we have $\frac{d}{dt}\dot{x} = -\omega^2 x$ Hamilton: $\dot{q} = p/m \wedge \dot{p} = -m\omega^2 q$

$\Leftrightarrow \ddot{x} + \omega^2 x = 0$ i.e $x(t) \propto \cos(\omega t) \vee \sin(\omega t)$

One can express state of the H.O. also via a single complex variabel:

$$\boxed{\alpha(t) = c \cdot \left(x(t) + \frac{i}{\omega} \dot{x}(t) \right)} \propto e^{-i\omega t}$$

COMPLEX $(\alpha \neq \alpha^*)$

where C is a constant. In this case:

$$\frac{\partial}{\partial t} \alpha(t) = \dot{\alpha}(t) = c\left(\dot{x} + \frac{i}{\omega}\ddot{x}(t)\right)$$
$$= c(\dot{x} - i\omega x) = -i\omega c\underbrace{\left(x + \frac{\dot{x}}{i\omega}\right)}_{\alpha}$$
$$\boxed{\partial_t \alpha = -i\omega \alpha(t)}$$

This new variabel can be used to express the energy of the H.O. as:

$$\boxed{E = \frac{m\omega^2}{4C^2}(\alpha^*\alpha + \alpha\alpha^*)}$$ C: constant

The quantization of the H.O. corresponds to: $\boxed{\frac{\hbar\omega}{2} = \frac{m\omega^2}{4C^2}}$

thus $$\boxed{\hat{H} = \frac{\hbar\omega}{2}(\hat{a}^+\hat{a} + \hat{a}\hat{a}^+)}$$ (note the symmetric form)

Therefor we notice the role of α and α^* which are replaced by $\alpha \to \hat{a}$ and $\alpha^* \to \hat{a}^+$. Similarly in quantum optics $\underbrace{\vec{E}}_{} \propto \alpha \to \hat{a}$ $E^* \propto \alpha^* \to \hat{a}^+$. $[a, a^+] = 1$ commutator relation

classical Electrical Field (Glauber).

Note that
$$\boxed{\begin{aligned} \hat{a} e^{-i\omega t} &= \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\hat{x} + i\hat{p}) \\ \hat{a}^+ e^{+i\omega t} &= \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\hat{x} - i\hat{p}) \end{aligned}}$$

or. $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^+)$ and $\hat{p} = \sqrt{2m\omega\hbar}\left(\frac{\hat{a} - \hat{a}^+}{i}\right)$

$$[\hat{x}, \hat{p}] = i\hbar$$

Review Quantized Harmonic Oscillator

Thus: $\hat{H} = \hat{p}^2/2m + \frac{1}{2} m\omega^2 \hat{x}^2$

$$\frac{d}{dt}\hat{x} = \left(\frac{1}{i\hbar}\right)[\hat{x},\hat{H}] = \hat{p}/m$$
$$\frac{d}{dt}\hat{p} = \left(\frac{1}{i\hbar}\right)[\hat{p},\hat{H}] = -m\omega^2\hat{x}$$

Heisenberg Equation of motion

Schrödinger Eq & wave function:

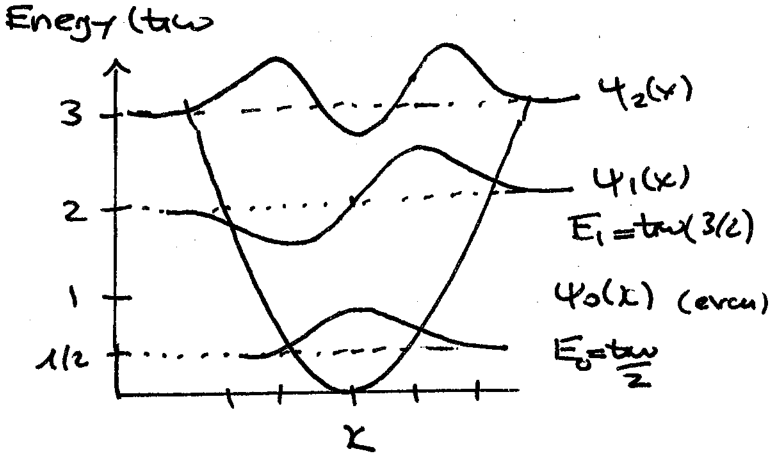
$$a|0\rangle = 0 \Rightarrow \frac{1}{\sqrt{2m\hbar\omega}}(\hat{p} - im\omega\hat{x})|0\rangle = 0 \quad | \langle x$$

$\psi_0 = \langle x|0\rangle$ and x is position basis $\{|x\rangle\}$

with $\langle x|\hat{p}|0\rangle = \frac{\hbar}{i}\frac{\partial}{\partial x}\psi_0$

thus: $\left(\frac{\hbar}{i}\frac{\partial}{\partial x} - im\omega x\right)\psi_0(x) = 0$

$$\Rightarrow \quad \psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar} \qquad \text{wave Fct.}$$



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Review:

Fock States (also called 'Number' States)

$$\hat{H} = \hbar\omega(\hat{a}^{+}\hat{a} + \tfrac{1}{2})$$

Thus the Eigenstates are :

$$\hat{H}|n\rangle = \hbar\omega(\hat{a}^{+}a + \tfrac{1}{2})|n\rangle = \cancel{\hbar\omega(n+)} \underbrace{E_n}_{\substack{\text{eigenenergy} \\ \text{(value)}}}|n\rangle \quad \Big| \times a \text{ from left}$$

use commutator $[\hat{a}, \hat{a}^{+}] = 1$

$$\hat{H}\underbrace{a|n\rangle}_{|n-1\rangle} = (E_n - \hbar\omega)\, a|n\rangle$$

Thus: $a|n\rangle$ is also eigenstate with $E_{n-1} = (E_n - \hbar\omega)$

Normalization requires $\langle n-1|n-1\rangle = 1$

repeat procedure: $a|0\rangle \overset{!}{=} 0$ ground state since

i.e. $\cancel{E a|0\rangle} = \underbrace{(E_0 - \hbar\omega)}_{\text{must be zero.}} a|0\rangle \Rightarrow \cancel{E_0 \ne \frac{\hbar\omega}{2}}$

$$\hat{H}|0\rangle = \frac{\hbar\omega}{2}|0\rangle \Rightarrow E_0 = \frac{\hbar\omega}{2}$$ Vacuum Energy

__Note:__ This engy leads to Casimir-force.

Quantum Optics I
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(Milburn Walls)

Quantization of the Electromagnetic Field

Maxwell Equations (no sources) $\qquad$ ρ: charges j: currents

$\vec{\nabla}\times\vec{E} = -\partial_t\vec{B}$ [Law of Induction]

$\vec{\nabla}\cdot\vec{B} = 0$ [no monopoles]

$\vec{\nabla}\cdot D = 0 (=\rho)$ [Gauss law ($\rho=0$, no charges)]

$\vec{\nabla}\times\vec{H} = \frac{\partial\vec{D}}{\partial t}(+j)$ [Biot-Savart law]

and $\vec{B}=\mu_0\vec{H}$ and $\vec{D}=\varepsilon_0\vec{E}$ (free space)

Using $\vec{\nabla}(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{E})-\vec{\nabla}^2E$

we obtain:

$$\boxed{\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} = 0}$$ (no sources)

Solution: Assume a box with dimensions L^3.

In 1-dimension

$$\vec{E} = \vec{e}_x\sum_j A_j$$

Introducing the vector potential $\vec{A}(r,t)$ in Coulomb guage

$(\vec{\nabla}\cdot\vec{A}=0)$ $\quad \left|\begin{array}{l}\vec{B}=\vec{\nabla}\times\vec{A}\\ \vec{E}=-\partial_t\vec{A}\end{array}\right|$

and: $\quad \nabla^2\vec{A}(r,t) - \frac{1}{c^2}\partial_t^2\vec{A}(r,t) = 0$

Solution: $\quad \vec{A}(r,t) = \sum_k c_k \underbrace{\vec{u}_k(r)}_{\text{vector}} e^{-i\omega_k t}$ $\quad \vec{\nabla}\cdot\vec{u}_k(r)=0$ ie transverse mode fct.

Assume that u_k is a normal set $\quad \int u_k(\vec{r})\,u_{k'}(\vec{r})\,dr = \delta_{kk'}$

(1) Traveling waves:

Example two metallic mirrors: $\vec{E}(0)=\vec{E}(L)$ = periodic B.C

$\Rightarrow k_x = \frac{2\pi}{L}n_x \quad k_y=\frac{2\pi}{L}n_y \quad k_z=\frac{2\pi}{L}n_z \ \wedge \ \vec{u}_k(r) = \cdot\, e^{+i\vec{k}\vec{r}}\cdot\vec{e}$

traveling Plane wave

$c|k|=\omega$

Quantization

The vector potential is written now

$$A(r,t) = \sum_k\left(\frac{\hbar}{2\omega\varepsilon_0 L^3}\right)^{1/2}[\hat{a}_k u_k(r)e^{-i\omega_k t} + \hat{a}^+ u_k^*(r)e^{+i\omega_k t})$$

The Field is correspondingly: (traveling wave)

$$\boxed{\vec{E}(r,t) = i\sum_k\left(\frac{\hbar\omega_k}{2\varepsilon_0 L^3}\right)^{1/2}[\hat{a}_k\vec{u}_k(r)e^{-i\omega_k t} - \hat{a}_k^+\vec{u}_k^*(r)e^{+i\omega_k t}]}$$

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traveling wave cont'd:

$$B(r,t) = i\sum_k \left(\frac{\hbar k_k}{2\varepsilon_0 L^3}\right)^{1/2} [\hat{a}_k(\vec{k}\times\vec{u}_k)e^{-i\omega_k t} + \hat{a}^+(\vec{k}\times\vec{u}_k)e^{+i\omega_k t})$$

L^3: volume

The Hamiltonian of the electromagnetic field is given by: (classical)

$$H = \frac{1}{2}\int(\varepsilon_0\vec{E}^2 + \mu_0\vec{H}^2)$$

Quantization now proceeds by first replacing the ~~operator~~ fields with operators: $\vec{E}\rightarrow\hat{\vec{E}}$ and $\vec{H}\rightarrow\hat{\vec{H}}$

and applying the SYMMETRIZATION postulate:

$$\hat{H} = \frac{1}{2}\int \varepsilon\, \vec{E}\cdot\vec{E}^* + \mu_0\vec{H}\vec{H}^* = \frac{1}{2}\int\frac{\varepsilon(EE^*+E^*E)}{2}+\ldots$$

This then yields:

$$\hat{H} = \frac{1}{2}\int \varepsilon\frac{(\hat{E}\hat{E}^+ + \hat{E}^+\hat{E})}{2} + \mu_0\frac{(\hat{H}\hat{H}^+ + \hat{H}^+\hat{H})}{2}$$

inserting yields (homework)

$$\boxed{\hat{H} = \sum_k \hbar\omega_k(\hat{a}_k^+\hat{a}_k + \tfrac{1}{2})}$$

$\hat{a}_k^+ a_k = n_k$

$n_k|n\rangle = n|n\rangle$ Fock state.

Reminder: compare result to classical Electrodynamics: (travel. waves),

$$\boxed{\begin{aligned} E(\vec{r},t) &= \sum_k \vec{\varepsilon}_k\, E_k^{(VAC)}\alpha_k(t)e^{-i\omega_k t + i\vec{k}\vec{r}} + c.c \\ H(\vec{r},t) &= \frac{1}{\mu_0}\sum_k \frac{\vec{k}\times\vec{\varepsilon}_k}{\omega_k}E_k^{(VAC)}\alpha_k(t)e^{-i\omega_k t + ik.r} + cc \end{aligned}}$$

Here α_k is classical component of vector potential. Define:

$$\mathcal{E}_k^{vac} \equiv \left(\frac{\hbar\omega_k}{2\varepsilon_0 V}\right)^{1/2} \quad \text{vacuum field.}$$

$$A = \sum_k \vec{\varepsilon}_k\, \mathcal{E}^{(VAC)}\cdot(\alpha_k(t)\vec{\varepsilon}\, e^{-i\omega_k t + i\vec{k}\vec{r}} + \alpha_k^*(t)\vec{\varepsilon}\, e^{+i\omega_k t - i\vec{k}\vec{r}})$$

Quantization proceeds by identifying $\alpha_k \rightarrow \hat{a}$, $\alpha_k^* \rightarrow \hat{a}^+$

and noting that $H_k = \int\frac{1}{2}\varepsilon|\vec{E}_k|^2 + \mu_0|\vec{H}_k|^2 = \frac{\hbar\omega_k}{2}(|\alpha_k|^2)$

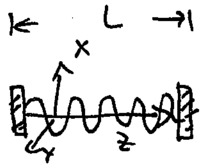
in this case.

Thus $H = \hbar\omega_k(\alpha_k\alpha_k^* + \alpha\alpha_k^*)$ classically.

$\Downarrow$

$\hat{H} = \hbar\omega_k(\hat{a}_k\hat{a}_k^+ + \hat{a}_k\hat{a}_k^+)$ quantum mechanically

Quantum Optics I



cavity field

Quantization inside a cavity:

Solution $\vec{A}(r,t) = \sum \alpha_k u_k(r) e^{-i\omega_k t}$ vector potential

$\vec{E}(r,t)$

CAVITY FIELD

and:
$$\boxed{\begin{aligned} E_x(z,t) &= \sum_k \left(E_k \alpha_k e^{-i\omega_k t} + E_k \alpha_k^* e^{+i\omega_k t}\right) \sin k_k z \\ H_y(z,t) &= \sum_k -i\varepsilon_0 c \left(E_k \alpha_k e^{-i\omega_k t} - E_k \alpha_k^* e^{+i\omega_k t}\right)\cos(k_k z) \end{aligned}}$$

and $E_k = \sqrt{\dfrac{\hbar\omega}{2\varepsilon_0 V}}$ vacuum field.

In this notation the energy takes a particularly simple form

$$H = \int \frac{1}{2}\varepsilon_0 |E|^2 + \frac{1}{2}\mu_0 |\vec{H}|^2 \, dV = \frac{\varepsilon_0}{2}\int dV \left(|\vec{E}|^2 + c^2 |B|^2\right)$$

$$= \int \frac{\hbar\omega}{2 V} \cdot \left(|\alpha_k|^2 \quad \right) dV = \frac{\hbar\omega}{2} \left(2|\alpha_k|^2 \cancel{}\right) = \frac{\hbar\omega}{2}\left(\alpha_k \alpha_k^* + \alpha_k^* \alpha_k\right)$$

The <u>quantization</u> now proceeds by setting $\boxed{\begin{aligned}\alpha_k &\to \hat{a}_k \\ \alpha_k^* &\to \hat{a}_k^+\end{aligned}}$

yielding $\boxed{\hat{H} = \frac{\hbar\omega}{2}\left(\hat{a}_k^+ a_k + \frac{1}{2}\right) = \hbar\omega a^+ a + \underbrace{\frac{\hbar\omega}{2}}_{\text{zero point energy.}}}$

This procedure can also be applied to any other H.O. e.g an L-C circuit, polaritons, ... etc the elementary excitations are quanta. ($\hbar\omega$).

Commutation relations:

$$\vec{E}(\vec{r},t) = \sum_k \vec{\varepsilon}_k^{(\lambda)} E_k \hat{a}_k e^{-i\omega_k t + \vec{k}\vec{r}} + c.c$$

$$\vec{H}(\vec{r},t) = \sum_k \frac{1}{\mu_0} \frac{\vec{k}\times\vec{\varepsilon}_k}{\omega_k} \vec{\varepsilon}_k^{(\lambda)} \hat{a}_k e^{-i\omega t + kr} + c.c$$

λ: polarization

ie
$$\alpha = A + \frac{i}{\omega}\dot{A}$$
$$\alpha^* = A - \frac{i}{\omega}\dot{A}$$

$$[a_{k,\lambda}, a_{k,\lambda'}] = 0 = [a^+_{k,\lambda}, \hat{a}^+_{k,\lambda}] \quad \text{and} \quad [\hat{a}_{k,\lambda}, \hat{a}^+_{k',\lambda'}] = \delta_{kk'}\delta_{\lambda\lambda'}$$

It follows that (homework, no proof):

$$[\hat{E}_j(\vec{r},t), \hat{H}_j(\vec{r}',t)] = 0$$ $\to$ one can measure E_x, H_x simultaneously.

BUT $[E_j(\vec{r},t), H_k(\vec{r}',t)] = -i\hbar c^2 \frac{\partial}{\partial \ell}\,\delta(\vec{r}-\vec{r}')$ j, k, l: cyclic permutation

$\Rightarrow$ perpendicular components cannot be measured simultaneously!

Quantum Optics I

$H = \mu_0 B = \frac{c^2}{\varepsilon_0} B$

Jackson check $S \to P$

$\frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c$

$\mu_0 = c^2 \cdot \frac{1}{\varepsilon_0}$

$\frac{1}{\mu_0} = c^2 \cdot \varepsilon_0$

<u>Momentum of Light</u> : poynting vector $\vec{S} = \vec{E} \times \vec{H} = \frac{1}{\ldots} \vec{E} \times \vec{B}\, \varepsilon$

$$P_{trans} = -\varepsilon_0 \int \vec{E}_k(r,t) \times \vec{B}_k(r,t)\, d^3_k \quad \text{(for transversal wave)}$$

$$= -\varepsilon_0 \int d^3k \cdot \underbrace{\dot{\vec{A}}}_{\vec{E}} \times \underbrace{(i\vec{k} \times \vec{A})}_{\vec{B}} \qquad \text{and } A = A(k,t)$$

$$P_{trans} = -\varepsilon_0 \int d^3k\, k \cdot \vec{A}^* \cdot A \qquad \text{MOMENTUM OF LIGHT}$$

Compute the operator $\hat{P}_{trans}$: $\alpha(k,t) \equiv \sqrt{\frac{\varepsilon_0 \omega}{2\hbar}} (A_\perp + \frac{i}{\omega} \dot{A})$

(u) apply symmetrisation

$$\hat{P}_{trans} = \varepsilon_0 \sum \frac{\omega^2}{4 \cdot (\frac{\varepsilon_0 \omega}{2\hbar})} \cdot [\alpha_k^* \alpha_k + \alpha_k \alpha_k^*] \quad \text{(Energy)}$$

$$P_{trans} = \varepsilon_0 \int d^3k \frac{\omega}{4 (\sqrt{\frac{\varepsilon_0 \omega}{2\hbar}})^2} k [\alpha_k^* \alpha_k + \alpha \alpha_k^*]$$

$$\boxed{\hat{P}_{trans} = \sum_k \frac{\hbar |\vec{k}_k|}{2} \cdot [\hat{a}_k^+ \hat{a}_k + \hat{a}_k \hat{a}_k^+] = \sum \hbar k_k \hat{a}_k^+ \hat{a}_k}$$

Very intuitive result.

$$\hat{P}_{trans} \underbrace{|n_k\rangle}_{\text{Fockstate}} = n_k \hbar k |n_k\rangle$$

i.e. <u>momentum</u> <u>multiples of $\hbar k$</u>.

vacuum state: $0\hbar k$ $|0\rangle$

(i.e. $\hat{a}|n_k\rangle = \sqrt{n}\, |n-1\rangle$
$\hat{a}^+|n_k\rangle = \sqrt{n_k+1}\, |n_{k+1}\rangle$)
$a|0\rangle = 0$

Vacuum Field of photons

$$\langle 0 | \vec{E}_k | 0 \rangle = \langle 0 | i \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} (\hat{a}_k e^{-i\omega t + i\vec{k}\vec{r}} \vec{\varepsilon} + \hat{a}_k^+ e^{+i\omega t - i\vec{k}\vec{r}} \vec{\varepsilon}) | 0 \rangle$$

$$\propto \langle 0 | a_k | 0 \rangle + \langle 0 | \hat{a}_k^+ | 0 \rangle = 0$$

But:

$$\langle 0 | \vec{E}_k^2 | 0 \rangle = \langle 0 | i \left(\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\right)^2 (\hat{a}_k e^{-i\omega t + ikr} + \hat{a}_k^+ e^{+i\omega t - ikr})^2 | 0 \rangle$$

$$= \left(\frac{\hbar\omega_k}{2\varepsilon_0 V}\right) \quad \text{thus } |\vec{E}_k| = \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}$$

Electrical vacuum field. (same for $\vec{B}^{vac}$) $B^{vac} = E^{vac}/c$

$\Rightarrow$ Vacuum Fluctuations (responsible for spontaneous emission)

compute $\langle n | \vec{E}_k^2 | n \rangle$. (homework)

Quantum Optics I

States of the Electromagnetic Field:

COHERENT STATES.

$a|\alpha\rangle = \alpha|\alpha\rangle$ thus $\langle\alpha|\hat{a}|\alpha\rangle = \alpha$

$\langle\alpha|\hat{a}^{+}|\alpha\rangle = \alpha^*$

Thus: $\langle\alpha|a^{+}a|\alpha\rangle = \alpha\cdot\alpha^* = |\alpha|^2$ i.e mean photon number of a coherent state.

Coherent states satisfy: $|\alpha\rangle = e^{-|\alpha|^2/2}\sum \frac{\alpha^n|n\rangle}{\sqrt{n!}}$

Thus: $p(n) = |\langle\alpha|n\rangle|^2 = \frac{|\alpha|^{2n}e^{-|\alpha|^2}}{n!}$ POISSON DISTRIBUTION

The coherent state is a displaced vacuum state:

$D(\alpha) \equiv \exp(\alpha\hat{a}^{+} - \alpha^* a)$

operator theorem in Q.M: $e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-[\hat{A},\hat{B}]/2}$

$$D(\alpha) = e^{-|\alpha|^2/2}e^{\alpha\hat{a}^{+}}e^{-\alpha^* a}$$

and $\boxed{|\alpha\rangle = D(\alpha)|0\rangle}$

Proof: $D^{+}(\alpha)\hat{a}|\alpha\rangle = D^{+}(\alpha)\hat{a}D(\alpha)|0\rangle = (\hat{a}+\alpha)|0\rangle = \alpha|0\rangle$

using the property: $D^{+}(\alpha) = D^{-1}(\alpha) = D(-\alpha)$

$D^{+}(\alpha)\hat{a}D(\alpha) = \hat{a}+\alpha$ (H.W)

Coherent states are not a Basis:

$$|\langle\alpha|\beta\rangle|^2 = \exp(-\tfrac{1}{2}(|\alpha|^2 + |\beta|^2) + \alpha\beta^*)$$
$$= \exp(|\alpha-\beta|^2)$$

Note that since a is non Hermetian $(a \neq \hat{a}^{+})$ the eigenvalue is COMPLEX.

The coherent states form an overcomplete basis:

$$\frac{1}{\pi}\int|\alpha\rangle\langle\alpha|\,d^2\alpha = \mathbb{1}$$